Novel Methods For Splicing Topoi Via Semantic Fibering Techniques

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Introduction

- Algorithm design $\cap$ category theory
- Goal: construct $f : \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}$
- Previously: splicing logics (fibering, fusion), stability of topos structure under certain operations (slice theorem, lex comonad theorem, lex idempotent monad thorem, three topos theorems in one) [1] [3] [4]

Background: Logic

- "Logic" – methods for [deductive] reasoning
- "***A*** Logic" – system consisting of **symbols**, **formal grammar** for WFFs, formal **evaluation rules**

	Syntactic	Semantic
Eval Description	{ sentence }* $\to$ sentence	{ sentence }* $\to$ truth value
Example	$\frac{\{X \vdash P \ , \ X \vdash Q\}}{X \vdash P \wedge Q}$ | $\frac{X \vdash P}{X \vdash P \vee Q}$	$\{P \wedge Q, Q, \neg P\} \mapsto F$
Example System	Sequent Calculus	Kripke semantics

Category Theory: Technical

- Category: $\mathbf{C} = \langle Ob(C), Hom(C), \circ \rangle$
  objects $A_1, \cdots, A_n \in Ob(C)$;
  morphisms $A_1 \xrightarrow{f_k} A_2 \in Hom(C)$;
  $id_A \in Hom(A, A)$ for each A;
  $\circ : A_1 \xrightarrow{f_k} A_2 \times A_2 \xrightarrow{f_k} A_3 \to A_1 \xrightarrow{f_k} A_3$ [$\circ$ is *associative*]
- Further fundamental concepts: 7 dogmas [2]
  categories, functors, naturality, limits, adjoints, colimits, comma categories;
  also **epis, monos, pullbacks, subobjects**, $\cdots$

Category Theory: Functionality

- Metamath: abstracts structure $\xrightarrow{\text{yields}}$ analogies
- Compositionality: barebones for associatively composing ("gluing") maps
- Extensibility: can add structural conditions, instantiate, "stack" / "layer" categories $\to$ higher categories (foundation for higher-order mereotopology)
- Internalization: can formulate mathematical objects & systems of logic within rich-enough categories (e.g. groups $\hookrightarrow$ finite product $\mathbf{C}$)

Topos Theory: Technical

- [Elementary] Topos $\mathcal{E}$: category with finite limits + power objects (finite limits + cartesian closed + subobject classifier)
- Internal logic is higher-order intuitionistic logic
- Internal language is MB type theory

Topos Theory: Functionality

- $\mathcal{E}$: Setting/foundation for mathematical reasoning (cf. its internal logic), alternative to set theory
- Has nice properties similar to **Set**
- Constructive mathematics

Splicing Logics

- **Algebraic Fibering**: coproduct [in e.g. the category of Hilbert calculi] of logics' signatures (*unconstrained fibering*); coproduct of logics' signatures followed by coequalizer (*constrained fibering*)

Mitchell-Benabou Type Theory

Internal Language of $\mathcal{E}$; $Ob(\mathcal{E}) \rightsquigarrow$ Sets

- types – $A \ \sim$ objects
- vars – $v : A \sim id_A$
- terms – $t(x_i : X_i) : B \sim \prod_i (X_i) \to B$
- formulas – $\varphi(x) \ \sim$ terms of type Ω
- connectives $\sim$ Ω's internal Heyting algebra structure
- quantifiers $\sim$ internal completeness of Ω
- truth of $\varphi \ \sim$ arrow interpreting P factors through $1 \xrightarrow{\text{true}} \Omega$

Interpreted via Kripke-Joyal semantics

Kripke-Joyal Semantics

- $U \vDash \varphi(\alpha) \wedge \psi(\alpha)$ iff $U \vDash \varphi(\alpha)$, $U \vDash \psi(\alpha)$
- $U \vDash \varphi(a) \vee \psi(a) i$ iff $\exists g : U_1 \to U, h : U_2 \to U$ s.t. $\langle g_1, g_2 \rangle : U1 \times U_2 \xrightarrow{epi} U, U_1 \vDash \varphi(a \circ g_1), U_2 \vDash \psi(a \circ g_2)$
- $U \vDash \varphi(a) \implies \psi(a)$ iff $\forall g : V \to U[V \vDash \varphi(a \circ g)]$ implies $V \vDash \psi(a \circ g)$
- $U \vDash \neg\varphi(a)$ iff $\forall g : V \to U[V \vDash \varphi(a \circ g)]$ implies $V \simeq 0$
- $U \vDash \exists y[\varphi(a, y)]$ iff $\exists g : V \xrightarrow{epi} , \ h : V \to Y[V \vDash \varphi(a \circ g, h)]$
- $U \vDash \forall y[\varphi(a, y)]$ iff $\forall g : V \xrightarrow{epi} U, h : V \to Y[V \vDash \varphi(a \circ g, h)]$

Algorithm Sketch

$\mathcal{E}_1 \to MB(\mathcal{E}_1) \to KJ(MB(\mathcal{E}_1))$
$\mathcal{E}_2 \to MB(\mathcal{E}_2) \to KJ(MB(\mathcal{E}_2))$
$\implies \mathcal{E}_{\text{free}}(\text{algfib}(KJ(MB(\mathcal{E}_1)), KJ(MB(\mathcal{E}_2)))$ (in the sense of [1])

Example

- $\mathcal{E}_1$ = category of simplicial sets ; $\mathcal{E}_2 = V_{\omega+1} \times V_{\omega+1}$
- Then, resulting $\mathcal{E}$ is a higher-order cellular automaton.

Future Work

- Extend to internal logics/languages of categories in general, & higher categories
- Explore applications to semantic web & data mining (particularly in worldview formalisms)

References

[1] (2008) Carnielli, W. et al. "Analysis and Synthesis of Logics: How to Cut and Paste Reasoning Systems". Springer.
[2] (1991) Joseph A. Goguen. "A Categorical Manifesto". Mathematical Structures in Computer Science. 49–67.
[3] https://ncatlab.org/toddtrimble/published/Three+topos+theorems+in+one
[4] (2003) Johnstone, P.T. "Sketches of an Elephant: A Topos Theory Compendium". Clarendon Press.